Numerical Integration

Purpose. The purpose of this project is to derive formulas for upper bounds on the amount of error you get from using a numerical method to approximate an integral. When you are finished with the project you should understand the error using Riemann sums and the trapezoid rule.

The story. Your friend Sue is an electrical engineering student who is trying to design a digital filter. Her professor told her to take the Gibbs phenomenon into account in her design. In order to do this, she needs values for:

$$\int_0^b \frac{\sin x}{x} dx$$

for $b = 0.10, 0.20, 0.30, 0.40, \cdots, 2.00$. Furthermore, she wants to be sure all the numbers she gets are accurate to within 0.000005. She knows that it is hopeless to find the antiderivative of $\frac{\sin x}{x}$. She that you are taking calculus and probably remember the formulas used to integrate functions numerically, so she asks for your help. Your job is to make a table of values for the above integral. You must be able to prove to Sue that your answers are within the accuracy that she needs.

Procedure. You are to follow the outline below. In the first 4 steps do not assume that the function is $f(x) = \frac{\sin x}{x}$. After you derive the appropriate formulas, then in step 5 you will apply what you learned to the special function under consideration. Before you start you may wish to look at the error bounds in the book using the Trapezoid rule and using Simpson's rule. The error bounds you derive will probably not be quite as good as those in the book, but they will be close.

1. Before you get started with the project you should be aware of a fact about integrals. Draw a picture to illustrate the inequality $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$ and explain why it is true. You need not give a proof, but you should give a careful explanation. This fact may be useful in later steps.
2. First you are to analyze how close a Riemann sum is to the actual integral. You are to assume that you have a function $f(x)$ whose derivative is continuous on the interval $[a, b]$. Furthermore, assume that $|f'(x)| \leq M$ for every $a \leq x \leq b$.
   a. Write the Riemann sum for $\int_a^b f(x)dx$ using the left end points.
   b. For each subinterval $[x_{i-1}, x_i]$ of the partition of $[a, b]$ consider the function $g(x) = f(x) - f(x_{i-1})$. First, note how the derivative of f and the derivative of g compare. Use an initial value problem to write the value of $g(x)$ in terms of an integral and $g(x_{i-1})$. Using this formula, give an upper bound for $|g(x)|$ when x is in the interval $[x_{i-1}, x_i]$.
   c. How does the integral of $g(x)$ relate to the error in approximating $\int_a^b f(x)dx$ with the Riemann sum using left end points? Use your estimate in part b) to derive an upper bound for the absolute value of the error on the interval $[x_{i-1}, x_i]$.
   d. Now use the error estimate you gave in part c) to derive an error estimate over the whole interval $[a, b]$. Your answer should involve a, b, M, and n, but not x_i or x_{i-1}.
   e. Try computing $\int_0^\pi \sin x dx$ using left end points for $n = 10, 20, 30, 40$ using a computer or calculator. Since you know the value of the integral, you can compare the actual value of the integral to the estimate you get from Riemann sums. Is your error estimate close to what you expect it to be? You may want to draw a graph to illustrate how close your estimates for the error are to the actual error in using the Riemann sum. Next, do the same for $\int_0^\pi \sin x dx$. Explain why in one case the error estimate is closer to the actual error than in the other case.
3. Next you will analyze how close the Trapezoid rule is to the actual value of the integral. The error you derive will not be as close to the actual error as the estimate in the book, but it will still give an estimate which is at least as large as the actual error. For this part we will assume that $f''(x)$ is continuous on the interval $[a, b]$.
   a. Let $T(f, n)$ denote the answer you get when you apply the Trapezoid rule to f using n subintervals. Show that if f and g are functions then $T(f + g, n) = T(f, n) + T(g, n)$ for any n.
   b. Let $E(f, n) = \int_a^b f(x)dx - T(f, n)$ denote the error from using the Trapezoid method in computing the integral of f. Show that if f is a linear function ($f(x) = mx + c$), then $E(f, n) = 0$. Also show that $E(f + g, n) = E(f, n) + E(g, n)$ and $E(cf, n) = cE(f, n)$ where c is a constant and f and g are continuous functions on $[a, b]$.

c. Write the Trapezoid rule and separate out the part corresponding to the i^{th} subinterval in the partition of $[a, b]$. You will estimate the error in each subinterval and then add your estimates together to get an estimate on the total error.
d. In the i^{th} subinterval, let $g(x)$ be the function whose graph is the tangent line to $f(x)$ at x_{i-1}.
e. Compute the first two derivatives of $h(x) = f(x) - g(x)$ and simplify as much as possible. In particular, compute $h(x_{i-1})$ and $h'(x_{i-1})$.
f. For the rest of 3) assume that $|f''(x)| \leq M$ for some constant M and every x in $[a, b]$. Use this assumption to estimate $|h'(x)|$ by using an initial value problem.
g. Now use your estimate in part f) to estimate $|h(x)|$, again using an initial value problem.
h. Estimate the integral of $|h|$ using your results in part g).
i. Estimate what you get using the Trapezoid rule on h for the subinterval $[x_{i-1}, x_i]$. Then combine this estimate with what you have in h) to get an estimate for the error using the Trapezoid rule on $[x_{i-1}, x_i]$.
j. Compare the error in using the Trapezoid rule on f and on h for the subinterval $[x_{i-1}, x_i]$ to estimate the error on f. Add up the error estimates on the subintervals to get an estimate for $E(f, n)$.
k. Do 2) part e)using the Trapezoid rule and integrating $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin x dx$ and $\int_0^{\frac{\pi}{2}} \sin x dx$. Which gives an error estimate closer to the actual error? Explain why.

4. Now consider the special function $f(x) = \frac{\sin x}{x}$.
   a. First you need an estimate of the appropriate derivatives of f. Use Mathematica, Maple, or your graphing calculator to estimate these from a graph. Note that at $x = 0$ there may be a slight problem. Does the graph show a problem? Explain.
   b. Now use the estimates you derived to decide what n should be to get the values of the integral within the desired error bounds for each of the three methods. Comment about which method is more efficient.
   c. Write a program (use Mathematica, Maple, C, Basic, or even a calculator) to compute the desired integrals. You may use whichever method you desire. (You may even use Simpson's method if you do the extra credit.)

If you haven't had enough fun yet, try deriving a formula for the error term using Simpson's formula. This is for extra credit, and is not required.