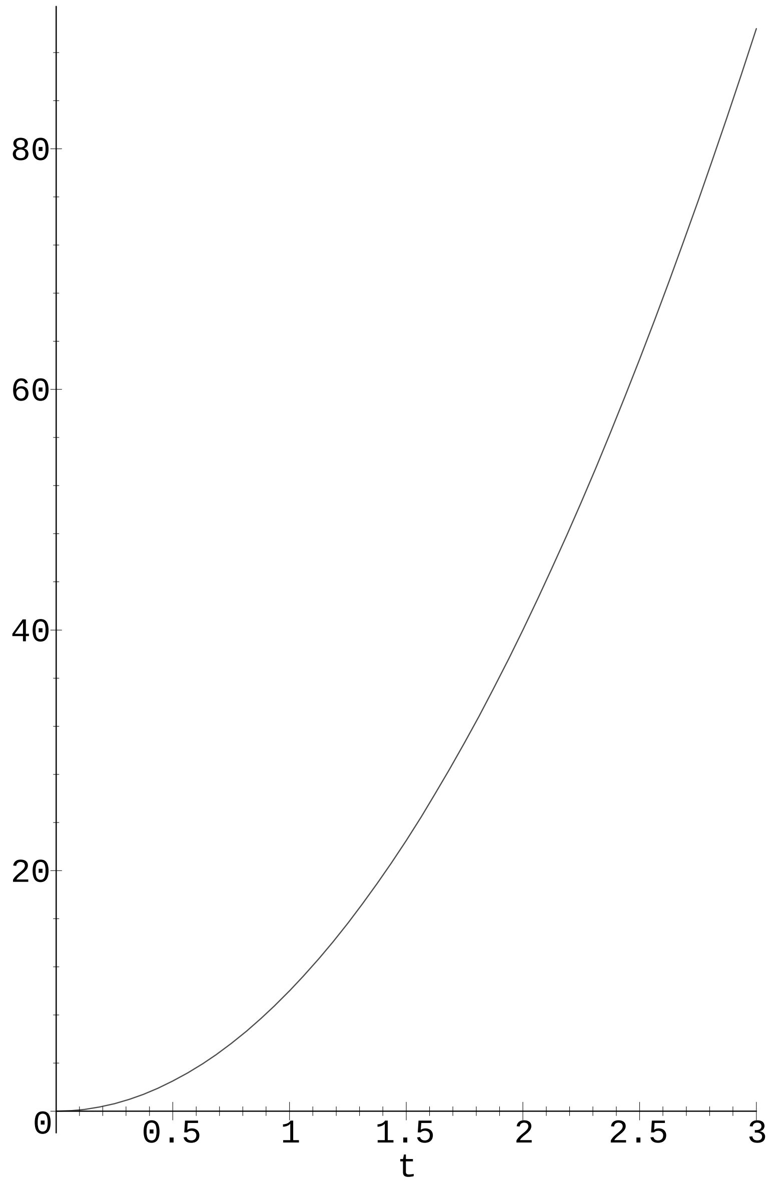
An Introduction to Derivatives

The purpose of this project is to give you an introduction to the ideas that we will be using in calculus. Two of the main concepts we will be learning this semester are the concept of a limit and the the concept of the derivative. This project will give you an introduction to both.

Old Stuff. A car at an intersection sits at a stop light. Once the light turns green, it accelerates away from the intersection. Assume that t seconds after the light turns green, the car is a distance of $f(t) = 10t^2$ meters away from the intersection. Then we can graph the distance from the intersection as a function of time.

How far is the car from the intersection after 2 seconds? 3 seconds? 2.5 seconds? 2.1 seconds?



New Idea. Let's now try to figure out how *fast* the car is moving after 2 seconds. Notice that we *can't* use the venerable distance-rate-time formula because the speed of the car is *changing* with time. However, we can *estimate* the speed at $t = 2$ seconds by dividing the distance it travels between 2 and 3 seconds by the time elapsed between 2 and 3 seconds:

$$\begin{aligned}\text{Approximate speed} &= \frac{f(3) - f(2)}{3\text{ s} - 2\text{ s}} \\ &= \frac{90\text{ m} - 40\text{ m}}{3\text{ s} - 2\text{ s}} \\ &= 50\text{ m/s}.\end{aligned}$$

(The symbol "m/s" of course means "meters per second.") Geometrically, this corresponds to the *slope of the line* which connects $(2, 40)$ and $(3, 90)$. This line is called a *secant* of the curve. (This definition makes sense: recall from geometry that a secant of a circle is a line which intersect the circle at two points.) ¿From the point-slope form of a line, this secant line has equation

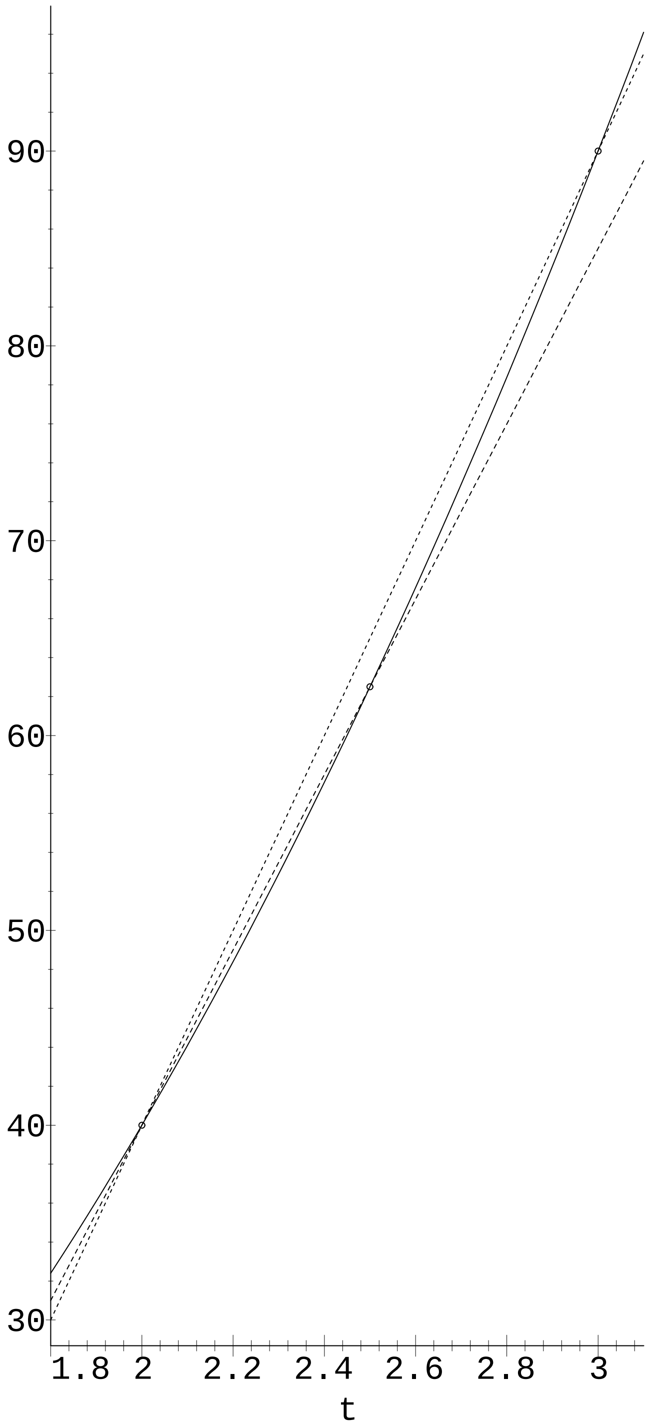
$$y = 40 + 50(t - 2).$$

However, the approximate speed calculated above isn't exactly correct, since the car does not maintain a constant speed between $t = 2$ seconds and $t = 3$ seconds. However, it's intuitively clear that we will get a *better* estimate of the speed by letting the time interval be *smaller*, since the speed of the car will not change as much over a smaller interval. Let's repeat the above calculation for a time interval which is say a half-second:

$$\begin{aligned}\text{Approximate speed} &= \frac{f(2.5) - f(2)}{2.5\text{ s} - 2\text{ s}} \\ &= \frac{62.5\text{ m} - 40\text{ m}}{0.5\text{ s}} \\ &= 45\text{ m/s}.\end{aligned}$$

Geometrically, this is the slope of the secant connecting $(2, 40)$ and $(2.5, 62.5)$; the equation of this line is

$$y = 40 + 45(t - 2).$$



Problem 1. The second approximation (45 m/s) is better than the first (50 m/s) since the time interval is smaller (a half second compared to one second). Let the length of this time interval be h. Fill in the following table for smaller and smaller h to obtain better and better approximations to the speed of the car two seconds after the stoplight turns green. The first two rows were computed on the previous page and are already filled in for you.

Approximate speed, when $t = 2$ s, of a car which travels $f(t) = 10t^2$ meters in t seconds

h	$2+h$	$f(2+h)$	$f(2+h) - f(2)$	Approximate speed
1 s	3 s	90 m	50 m	50 m/s
0.5 s	2.5 s	62.5 m	22.5 m	45 m/s
0.2 s				
0.1 s				
0.01 s				
0.0001 s				

Problem 2. Notice that these approximate speeds seem to converging to a number; this number is the true speed of the car after two seconds. Guess what the *exact* speed of the car will be at $t = 2$ seconds, and verify your guess. *Hint*: Simplify

$$\frac{f(2+h) - f(2)}{h} = \frac{10(2+h)^2 - 10(2)^2}{h},$$

and then (after simplifying) let h, the length of the time interval, tend to zero.

Problem 3. Determine the equations for the four new secant lines of Problem 1. Sketch these secant lines and the graph of $f(t) = 10t^2$; i.e., draw *one* figure that contains the four new secant lines. Draw this figure as accurately as you can. (If you're using a graphics calculator, get started by letting the t-axis have range $(1.9, 2.2)$ and the y-axis have range $(38, 50)$. You may wish to alter these ranges if you feel you can get a better picture.) Convince yourself that the slopes of these secant lines approaches the slope of a line which is *tangent* to $y = 10t^2$ at $(2, 40)$. Therefore, using the result of Problem 2, use the point-slope form of a line to find the equation of this tangent line.

Problem 4. In Problems 1 and 2, we calculated the speed of the car two seconds after the car turned green. Now repeat the above steps to find the speed of the car 1 second after the stoplight turns green. Repeat for $t = 3$ seconds. Repeat for $t = 4$ seconds.

Problem 5. Based on your answers to Problem 4, try to guess a pattern that will give you the speed of the car after t seconds. Prove that your formula actually works. *Hint:* Simplify $\frac{f(t+h)-f(t)}{h}$, as before, and then let h tend to zero (*after* you have simplified).

Problem 6. *Open Exploration.* Now apply this procedure to functions other than $f(t) = 10t^2$. Suggestions:

- $f(t) = t$
- $f(t) = t^2$
- $f(t) = t^3$
- $f(t) = t^4$
- $f(t) = 6t^3$ (compare your answer with the answer for $f(t) = t^3$)
- $f(t) = t + t^2$ (compare your answer with the answers for $f(t) = t$ and $f(t) = t^2$)

This next group of functions is somewhat more difficult; don't try them until you've finished the previous six.

- $f(t) = \sqrt{t}$
- $f(t) = \sin t$, where t is in radians (try to find the speed when $t = 0, \pi/6, \pi/4, \pi/3, \pi/2$)
- $f(t) = \cos t$, where t is in radians (try to find the speed when $t = 0, \pi/6, \pi/4, \pi/3, \pi/2$)
- $f(t) = e^t$
- $f(t) = \ln t$

The basic problem is the same for each of these functions: assume that a car travels $f(t)$ meters in t seconds, and guess a formula for how *fast* the car will be traveling in t seconds. If you can, prove your guesses. Carefully label the tables (similar to Problem 1) that you create with both the function $f(t)$ and the time t; you will soon be deluged with these tables and it will be easy to get mixed up. Always draw pictures of $f(t)$, its secants and its tangents to develop your intuition.