Derivatives Without Limits

Zack, a junior TAMS student, is taking Calculus 1710. He mentioned to his friend Liz, a junior TAMSter taking precalculus, that in his calculus class they were just starting to learn limits and then they would learn derivatives. Zach stated "When you get to calculus you will learn how to compute the slope of the tangent line to parabolas and other curves, but of course you first have to learn limits." Not to be shown up, Liz decides to investigate how to compute the slope of tangent lines to parabolas and other polynomials of higher degree. Your job is to help Liz discover the formula for finding the slope of tangent lines to polynomials - without using calculus concepts!

Liz will use a slightly different approach to derivatives than we will use in class. Her approach will be to consider double roots of polynomials. Let $p(x)$ be a polynomial. A root of $p(x)$ is a number a such that $p(a) = 0$. This is equivalent to saying that $x - a$ is a factor of $p(x)$. Equivalently, we can say that a is a root of $p(x)$ if $p(x) = (x - a)q(x)$ where $q(x)$ is a polynomial. For example, $a = 2$ is a root of $p(x) = x^2 - 3x + 2$ because $p(2) = 0$. Another way to see that 2 is a root is to factor: $p(x) = (x - 2)(x - 1)$.

A double root to $p(x)$ is a number a such that $(x-a)^2$ is a factor of $p(x)$. For example, 0 is a double root of $p(x) = x^3 + 2x^2$ since $p(x) = x^2(x + 2)$. Similarly, -1 is a double root of $x^3 + 2x^2 + x$, since $x^3 + 2x^2 + x = (x+1)^2x$. Another way to think of double roots is to combine the two ways of looking at roots. The polynomial $p(x)$ has a double root at $x = a$ if we can write $p(x) = (x - a)q(x)$ for some polynomial $q(x)$ and $q(a) = 0$. Do you see why this formulation of a double root gives you two factors of $(x - a)$ in $p(x)$? To use the example $x^3 + 2x^2 + x$, we can write $x^3 + 2x^2 + x = (x+1)(x^2 + x)$. Here, $q(x) = x^2 + x$, and $q(-1) = (-1)^2 + (-1) = 0$.

Suppose that x is very close to the number a, a double root of $p(x)$. Then $x - a$ is small, but $(x - a)^2$ is even closer to zero than $x - a$. For example, 0 is a double root of x^2 and the graph of $y = x^2$ looks pretty flat near the point where $x = 0$. This is the main idea behind how you should think about the tangent line to a graph in this project.

Given a polynomial $p(x)$, we say that the line $y = mx + b$ is a tangent line to $p(x)$ at the point a if the polynomial $p(x) - (mx + b)$ has a double root at $x = a$. For example, consider the function $p(x) = x^2$. How would you find the tangent line at the point where $x = 1$? Here is one way to do it. Consider an arbitrary line $y = mx + b$. In order to be the tangent line to x^2 at the point where $x = 1$, the line $y = mx + b$ must pass through the point $(1, 1)$ (since $(1, 1)$ is the point on $y = x^2$ where $x = 1$). According to the point-slope form, $y - 1 = m(x - 1)$, is the form of the equation of the tangent line. Simplified this becomes $y = mx - m + 1$. Now to be a tangent line at the point where $x = 1$, we need a double root of $x^2 - (mx - m + 1)$ at the point $x = 1$. We can simplify and factor $x^2 - (mx - m + 1) = x^2 - 1 - m(x - 1) = (x + 1)(x - 1) - m(x - 1) = (x + 1 - m)(x - 1)$. Letting $q(x) = x + 1 - m$, $x^2 - (mx - m + 1)$ has a double root at $x = 1$ exactly if $q(1) = 0$. But this means $0 = q(1) = 1 + 1 - m$, which says $m = 2$. Therefore, the line $y = 2x - 1$ is tangent to the graph of $y = x^2$ at the point where $x = 1$. The reason this is a reasonable definition for the tangent line is that the tangent line should be very close to the curve at the point of tangency. This means that the y-values on the graph of $p(x)$ and on the line should be very close together when x is close to a. A double root for the difference $p(x) - (mx + b)$ assures this.

Up to this point, you have been given background ideas and definitions. Here is where your work begins. Your goal is to derive the formula for the slope of the tangent line to the polynomial $p(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$. Remember, that you are not to use limits, derivatives or other ideas from calculus. Instead you are to use the idea of double roots and just things you know from precalculus.

First start by finding the tangent lines to $p(x) = x^2$ at the points $x = 3$, $x = -2$, and $x = 4$. Now use the same process to find the slope of the tangent line to $p(x) = x^2$ at a point $x = a$. In other words, instead of using a specific number, use a to represent an arbitrary number. The process works the same as it did with the examples you worked out.

What is the tangent line to $y = x$ at the point (a, a)? Use the double root definition above to determine this. Also, what is the tangent line to $y = c$ where c is a constant?

Do the same thing with the function x^3 as you did with x^2 above. Your goal is to find the slope of the tangent line to $y = x^3$ at the point on the graph with x-coordinate given by a, that is at the point (a, a^3). Then do x^4. Then do x^n for an arbitrary positive integer n. You are to derive a formula for the slope of the tangent line to $y = x^n$ at an arbitrary point (a, a^n) on the graph for every positive integer n.

To finish, you need to prove the following theorems:

Theorem 1 : *If $x = a$ is a double root for both polynomials $p(x)$ and $f(x)$, then $x = a$ is a double root for $p(x) + f(x)$ and for $cf(x)$ for any constant c.*

Use this theorem to prove the next theorem:

Theorem 2 : *If the tangent line to $p(x)$ at $x = a$ has slope m_1 and the tangent line to the $f(x)$ at $x = a$ has slope m_2, then the slope of the tangent line to $p(x) + f(x)$ at $x = a$ is $m_1 + m_2$. Furthermore, for any constant c, the slope of the tangent line to $cp(x)$ at $x = a$ is cm_1.*

Now, use what you learned about the slopes of the tangent lines to $x, x^2, x^3, \ldots, x^n$ to derive the formula for the slope of the tangent line to $p(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$.

Use the formula you derived to find the equation of the tangent line to $y = 4x^3 + 2x^2 - 6x + 3$ at the point where $x = 2$.