Integrating Polynomials

Purpose. The purpose of this project is to derive the formula for $\int_0^a x^m\,dx$ where k is a positive integer without using the Fundamental Theorem of Calculus. (No, you will not have to use induction - very much!)

The story. Your friend Les Bag does not believe the definition of the Riemann integral. He says, "The value of a Riemann sum depends on the partition and the choice of points c_k in the subintervals. Furthermore, every example we see in class or in the book uses equally spaced intervals. Maybe the choice of c_k does not matter since Dr. Brand showed an example in class that illustrated this, but still, what about different choices for the partition. I would like to see an example where the partition was not equally spaced, but the limit of the Riemann sum comes out the same as the limit using an equally spaced partition. I think there must be a better way to define an integral!"

Your job is to work through the details for a particular partition that is not equally spaced and compute $\int_0^a x^m\,dx$ for all positive integers m and all $a > 0$.

Procedure. You are to follow the outline below.

1. The first step is to recall the formula for the sum of a geometric series. The formula is $\sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}$. Prove this formula. You may use induction or any other method you wish.
2. Use the formula in part 1) to find a way to factor $1 - r^n$.

For the rest of the project, fix an arbitrary positive number a and an arbitrary integer $m > 0$.

3. Consider the partition P of $[0, a]$ with the points $0 < ar^{n-1} < ar^{n-2} < ar^{n-3} < \ldots < ar^2 < ar < a$ where $0 < r < 1$. What is $||P||$? Write a formula for the Riemann sum R_P using right end points. (It may be easier to write the sum backwards rather than the usual way it is written. In other words, start with the right most rectangle instead of the left most.)
4. Simplify the Riemann sum R_P you obtained. Part 1) may be helpful.
5. To compute the limit of the Riemann sum, you need to compute $\lim_{||P||\to 0} R_P$. Using the norm of the partition P that you found in Part 3), what limits must approach 0 as $||P|| \to 0$?
6. Find $\int_0^a x^m\,dx$ by computing the limit of the expression in part 4) (as $||P|| \to 0$). Use the conditions given in 5) in order to obtain your answer.
7. Explain how this example illustrates the Fundamental Theorem of Calculus and in what sense the partition does not matter when computing integrals.