A Chubby Chef's Polynomial

Purpose. The purpose of this project is to introduce you to an interesting set of polynomials and have you derive some of their basic properties.

The story. A chubby chef has just perfected a new way to make doughnuts. He is in the process of writing up his recipe and realizes that the time to fry his doughnuts depends on the size of the doughnuts. The perfect fry time is a polynomial function of the doughnut mass. The way he calculates it is by first taking the mass of the doughnut, subtracting an ideal doughnut mass, say m_0, and then scale to obtain a number x with $-1 \le x \le 1$. He then computes $f(x) = \frac{1}{2}x^5 + 3x^4 - x^3 + x^2 + 2x + 1$ to give the required frying time in some obscure units. His problem is that he knows that most doughnut makers would have trouble computing a fifth degree polynomial. So, he wishes to find a polynomial of degree 4, say $g(x)$, with the property that for any $-1 \le x \le 1$, $|f(x) - g(x)|$ is very small. For a polynomial g of degree 4, let E_g be the maximum value of $|f(x) - g(x)|$ for $-1 \le x \le 1$. The chubby chef wishes to find a g with E_g as small as possible. He wants the error E_g to be no more than 0.03. Is this possible?

After thinking about the problem for a while, he remembered that you are taking calculus. So naturally, he goes to you for help on the problem.

The procedure. Do the steps below.

1. First try guessing some polynomials $g(x)$ to see if you can find one that makes E_g small. After you are convinced that it will be hard to find the best possible polynomial $g(x)$, proceed on to step 2).
2. Although it may seem like trigonometry and polynomials do not have much to do with each other, read Section 6.8 in your book concerning the inverse trigonometric functions. Concentrate mainly on the inverse cosine function. You may wish to do a few problems at the end of the section to help you understand the idea of the arc cosine function.
3. Now define $T_n(x) = \cos(n \cos^{-1} x)$ for each non-negative integer n. Note that the domain of the function T_n is $[-1, 1]$. (Here, $\cos^{-1} x$ is used to represent the inverse cosine function.) Use this definition to rewrite $T_0(x)$, $T_1(x)$, and $T_2(x)$ in such a way that no trigonometric functions are involved.
4. Start with the formulas for $\cos(\alpha + \beta)$ and $\cos(\alpha - \beta)$ and derive a formula for the product $\cos(\alpha)\cos(\beta)$. Now be clever and use the formula for the product of cosines to find a relation between $T_{n-1}(x)$, $T_n(x)$ and $T_{n+1}(x)$. Solve it for T_{n+1}.
5. Use the relation you found in part 4) to find $T_3(x)$, $T_4(x)$, and $T_5(x)$. Plot their graphs. You may sketch them or use Mathematica to graph them.
6. Prove for each $n \ge 1$, $T_n(x)$ is a polynomial of degree n with leading coefficient 2^{n-1}. (Hint: use induction.)
7. $T_n(x)$ has exactly n root between -1 and 1. What are the roots? (Hint: Use the original definition.)
8. Find all the points where $T_n(x) = 1$ and all the points where $T_n(x) = -1$.
9. On the same axes draw the graph of $T_5(x)$ and a continuous function, $p(x)$, with domain $[-1, 1]$ and having the property that $|p(x)| < 1$ for $-1 \le x \le 1$. At how

many points do the graphs of the functions intersect? Make a conjecture and prove it concerning a continuous function $p(x)$ with the properties given above and $T_5(x)$. (Hint: Consider the difference $T_5(x) - p(x)$ and look for sign changes.)

10. Now, generalize your conjecture in part 9) to include all values of n. Then prove your conjecture.
11. Suppose that $p(x)$ is a polynomial of degree n with leading coefficient 2^{n-1}. What is the degree of the polynomial $q(x) = p(x) - T_n(x)$? What is the maximum number of roots that $q(x)$ can have? Assuming that $|p(x)| < 1$ for all $-1 \leq x \leq 1$, what does part 10) say about the number of roots of q? What is your conclusion?
12. Now, what does all this have to do with the original problem? Using the Chubby Chef's notation, write $p(x) = f(x) - g(x)$. If you want to find a $g(x)$ that gives a minimum error E_g, then you could equivalently find a polynomial $p(x)$ with the same leading coefficient as $f(x)$ so the maximum value of $p(x)$ for $-1 \leq x \leq 1$ is as small as possible. Use this observation to figure out what $p(x)$ should be and then determine $g(x)$. (Hint: Think about part 11) and multiply by a constant to make the leading coefficient what you want it to be.)
13. Based on what you have done above, determine E_g for the function g of part 12). Verify this by plotting the graph of $f(x) - g(x)$ for $-1 \leq x \leq 1$. What polynomial should the Chubby Chef use?
14. Often to approximate $\sin x$ for x small, the polynomial $f(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$ is used. Use the methods you developed in the previous steps to find a polynomial of degree less than 7 that is also a good approximation for $\sin x$ when x is small. Use mathematica to graph the difference between $\sin x$ and $f(x)$ and the difference between $\sin x$ and your approximation to $f(x)$.